

# Covering a 3-graph with perfect matchings

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November 15, 2011

## Abstract

Let  $G$  be a bridgeless cubic graph. A well-known conjecture of Berge and Fulkerson can be stated as follows: there exist five perfect matchings of  $G$  such that each edge of  $G$  is contained in at least one of them. Here, we prove that in each bridgeless cubic graph there exist five perfect matchings covering a portion of the edges at least equal to  $\frac{215}{231}$ . By a generalization of this result, we decrease the best known upper bound, expressed in terms of the size of the graph, for the number of perfect matchings needed to cover the edge-set of  $G$ .

*Keywords:* Berge-Fulkerson conjecture, perfect matchings, cubic graphs.  
*MSC(2010):* 05C15 (05C70)

## 1 Introduction

Throughout this paper, a graph  $G$  always means a simple connected finite graph (without loops and parallel edges). A perfect matching of  $G$  is a 1-regular spanning subgraph of  $G$ . As usual a bridgeless cubic graph will be called a 3-graph. Following the definition introduced in [5] and [9] we define  $m_t(G)$  as the maximum fraction of edges in  $G$  that can be covered by  $t$  perfect matchings, and by  $m_t$  the infimum of all  $m_t(G)$  over all 3-graphs. More precisely:

$$m_t = \inf_G \max_{M_1, \dots, M_t} \frac{|\cup_{i=1}^t M_i|}{|E(G)|}$$

The Berge-Fulkerson conjecture, one of the challenging open problems in graph theory, can be easily stated in terms of  $m_t$ : it is in fact equivalent to the assertion that  $m_5 = 1$  (see [7]).

Kaiser, Král and Norine furnish in [5] the exact value of  $m_2$  and they prove that  $m_3$  is at least  $\frac{27}{35}$ . In a final remark of their paper, they announced, without a proof, a general lower bound for  $m_t$ . In the present paper we furnish a proof of such a lower bound, and we deduce a new upper bound in term of  $t$  for the size of a 3-graph admitting a covering with  $t$  perfect matchings. More precisely, it

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was known (see for instance [9]) that a 3-graph of size  $E < (\frac{3}{2})^t$  can be covered using  $t$  perfect matchings. We improve this bound proving that each 3-graph with size  $E < \frac{2^t}{\sqrt{t}}$  can be covered with  $t$  perfect matchings.

Finally, as a by-product of our main result, we also obtain that in each 3-graph there exist a set of  $t$  perfect matchings with no  $(2t+1)$ -cut in their intersection, giving a partial support to a conjecture of Kaiser and Raspaud, [4], in a special form due to Mácajová and Skoviera, [6], about the existence of two perfect matchings with no odd cut in their intersection.

The main tool for our proof is the Perfect Matching Polytope Theorem of Edmonds, we briefly recall it in the next section.

## 2 The perfect matching polytope

Let  $G$  be a graph. A *cut*  $C$  in  $G$  is a subset of  $E(G)$  such that  $G \setminus C$  has more components than  $G$  does, and  $C$  is inclusion-wise minimal with this property. A  $k$ -cut is a cut of cardinality  $k$ . Let  $X$  be a subset of the vertex-set, we denote by  $\partial X$  the set of the edges with precisely one end in  $X$ .

Let  $w$  be a vector in  $\mathbb{R}^E$ . The entry of  $w$  corresponding to an edge  $e$  is denoted by  $w(e)$ , and for  $A \subseteq E(G)$ , we define the weight  $w(A)$  of  $A$  as  $\sum_{e \in A} w(e)$ . The vector  $w$  is said to be a *fractional perfect matching* of  $G$  if it satisfies the following properties:

- a)  $0 \leq w(e) \leq 1$ , for each  $e \in E(G)$ ,
- b)  $w(\partial\{v\}) = 1$  for each vertex  $v \in V$ , and
- c)  $w(\partial X) \geq 1$  for each  $X \subseteq V(G)$  of odd cardinality.

We will denote by  $P(G)$  the set of all fractional perfect matchings of  $G$ . If  $M$  is a perfect matching, then the characteristic vector  $\chi^M \in \mathbb{R}^E$  of  $M$  is contained in  $P(G)$ . Furthermore, if  $w_1, \dots, w_n \in P(G)$ , then any convex combination  $\sum_{i=1}^n \alpha_i w_i$  also belong to  $P(G)$ . It follows that  $P(G)$  contains the convex hull of all vectors  $\chi^M$  where  $M$  is a perfect matching of  $G$ . The Perfect Matching Polytope Theorem of Edmonds asserts that the converse inclusion also holds:

**Theorem 2.1** (Edmonds). *For any graph  $G$ , the set  $P(G)$  coincides with the convex hull of the characteristic vectors of perfect matchings of  $G$ .*

The main tool in our proof is the following property of a fractional perfect matching:

**Lemma 2.1.** *If  $w$  is a fractional perfect matching in a graph  $G$  and  $c \in \mathbb{R}^E$ , then  $G$  has a perfect matching  $M$  such that*

$$c \cdot \chi^M \geq c \cdot w$$

*where  $\cdot$  denotes the scalar product. Moreover, there exists such a perfect matching  $M$  that contains exactly one edge of each odd cut  $C$  with  $w(C) = 1$ .*

A proof of this lemma can be found in [5].

### 3 A lower bound for $m_t$

In this section we prove  $m_t \geq a_t$  for each index  $t$ , where  $a_t$  is the sequence defined by the law  $a_t = \frac{t}{2t+1}(1 - a_{t-1}) + a_{t-1}$  and  $a_0 = 0$ . In particular, we deduce  $m_5 \geq \frac{215}{231}$ .

Let  $\mathcal{M}^t = \{M_1, \dots, M_t\}$  be a set of  $t$  perfect matchings of a 3-graph  $G$ , and set  $\mathcal{M}^0 = \emptyset$ .

For each subset  $A$  of the edge-set of  $G$  we define

$$\Phi(A, \mathcal{M}^t) = \sum_{i=1}^t |A \cap M_i|.$$

Furthermore, we define the weight  $w_{\mathcal{M}^t}$  as:

$$w_{\mathcal{M}^t}(e) = \frac{t+1 - \sum_{i=1}^t |M_i \cap \{e\}|}{2t+3}.$$

Note that if  $|\mathcal{M}| = 1$  ( $= 2$ ), we obtain as particular case  $w_2$  ( $w_3$ ) defined in the proof of the main theorem of [5].

Let  $A$  be a subset of cardinality  $k$  of  $E(G)$ , the weight of  $A$  is given by the following relation:

$$w_{\mathcal{M}^t}(A) = \frac{k(t+1) - \Phi(A, \mathcal{M}^t)}{2t+3}.$$

In other words, by the very definition of the weight  $w_{\mathcal{M}^t}$ , the weight of a set just depends by the cardinality of the intersection with the perfect matchings in  $\mathcal{M}^t$ .

An easy calculation proves the following lemma,

**Lemma 3.1.** *Let  $A \subseteq E(G)$  be such that  $|A| = k$ .*

$$w_{\mathcal{M}^t}(A) \geq 1 \iff \Phi(A, \mathcal{M}^t) \leq t(k-2) + (k-3)$$

*and equality holds on one side of the implication if and only if it holds on the other side.*

We are now ready to state our main result.

**Theorem 3.1.** *Let  $a_t$  be the sequence defined recursively as follows:  $a_0 = 0$  and  $a_{t+1} = \frac{t}{2t+1}(1 - a_t) + a_t$ . Then,  $m_t \geq a_t$  for each index  $t$ .*

*Proof.* Let  $\mathcal{M}^t = \{M_1, \dots, M_t\}$  be a set of  $t$  perfect matchings such that  $w_{\mathcal{M}^t}$  is a fractional perfect matching of a 3-graph  $G$ . Set  $c_t = 1 - \chi^{\cup_{i=1}^t M_i}$ . By Lemma 2.1 there exists a perfect matching  $M_{t+1}$  such that

$$c_t \cdot \chi^{M_{t+1}} \geq c_t \cdot w_{\mathcal{M}^t}$$

and  $M_{t+1}$  contains exactly one edge for each cut  $C$  with  $w_{\mathcal{M}^t}(C) = 1$ . Now, we prove that  $w_{\mathcal{M}^{t+1}}$ , where  $\mathcal{M}^{t+1} = \mathcal{M}^t \cup \{M_{t+1}\}$ , is a fractional perfect matching of  $G$ . We have to verify the properties a), b) and c) of the definition of fractional perfect matching.

- a)  $0 \leq w_{\mathcal{M}^{t+1}}(e) \leq 1$  trivially holds for each  $e \in E(G)$ .  
b) Let  $v$  be a vertex of  $G$ .

$$w_{\mathcal{M}^{t+1}}(\partial\{v\}) = \frac{3(t+2) - \Phi(\partial\{v\}, \mathcal{M}^{t+1})}{2t+5}$$

since each perfect matching  $M_i$  intersects  $\partial\{v\}$  exactly one time we have  $\Phi(\partial\{v\}, \mathcal{M}^{t+1}) = t+1$  and then

$$w_{\mathcal{M}^{t+1}}(\partial\{v\}) = \frac{3(t+2) - (t+1)}{2t+5} = 1.$$

- c) Let  $C$  be a  $k$ -cut of  $G$ , with  $k$  odd.

$$w_{\mathcal{M}^{t+1}}(C) = \frac{k(t+2) - \Phi(C, \mathcal{M}^{t+1})}{2t+5}$$

If  $k = 3$ , by the assumption that  $w_{\mathcal{M}^t}$  is a fractional perfect matching, we have

$$w_{\mathcal{M}^t}(C) = \frac{3(t+1) - \Phi(C, \mathcal{M}^t)}{2t+3} \geq 1$$

that is  $\Phi(C, \mathcal{M}^t) \leq t = |\mathcal{M}^t|$ . On the other side a 3-cut intersects each  $M_i$  at least one time  $\Phi(C, \mathcal{M}^t) \geq t$ , then  $\Phi(C, \mathcal{M}^t) = t$  and  $w_{\mathcal{M}^t}(C) = 1$ . By Lemma 2.1, we obtain  $|C \cap M_{t+1}| = 1$ . Now, we can compute  $\Phi(C, \mathcal{M}^{t+1}) = \Phi(C, \mathcal{M}^t) + |C \cap M_{t+1}| = t+1$ . We have proved that

$$w_{\mathcal{M}^{t+1}}(C) = \frac{3(t+2) - \Phi(C, \mathcal{M}^{t+1})}{2t+5} = \frac{3(t+2) - (t+1)}{2t+5} = 1$$

as required.

If  $k > 3$ , we distinguish two cases according that  $w_{\mathcal{M}^t}(C) = 1$  (that is  $\Phi(C, \mathcal{M}^t) = t(k-2) + (k-3)$ ) or  $w_{\mathcal{M}^t}(C) > 1$  (that is  $\Phi(C, \mathcal{M}^t) < t(k-2) + (k-3)$ ).

In the former case we have  $|C \cap M_{t+1}| = 1$ , hence

$$\Phi(C, \mathcal{M}^{t+1}) = \Phi(C, \mathcal{M}^t) + |C \cap M_{t+1}| = t(k-2) + (k-3) + 1 = (t+1)(k-2) \leq (t+1)(k-2) + (k-3)$$

then  $w_{\mathcal{M}^{t+1}}(C) \geq 1$  by Lemma 3.1.

In the latter case we have  $|M_{t+1} \cap C| \leq k$ , hence

$$\Phi(C, \mathcal{M}^{t+1}) = \Phi(C, \mathcal{M}^t) + |C \cap M_{t+1}| < t(k-2) + (k-3) + k$$

since each perfect matching intersects an odd cut an odd number of times it follows  $\Phi(C, \mathcal{M}^t) \equiv t \pmod{2}$ , so the previous is equivalent to

$$\Phi(C, \mathcal{M}^{t+1}) \leq (t+1)(k-2) + (k-5) + k = (t+1)(k-2) + (k-3).$$

By Lemma 3.1, we can infer  $w_{\mathcal{M}^{t+1}}(C) \geq 1$  also in this case.

By induction, since the basic step  $w_{\mathcal{M}^0}(e) = \frac{1}{3}$  is trivially a fractional perfect matching, we have that  $w_{\mathcal{M}^t}$  (with  $\mathcal{M}^t$  constructed as described above) is a fractional perfect matching for each value of  $t$ .

Therefore, the following holds:

$$c_t \cdot \chi^{M_{t+1}} \geq c_t \cdot w_{\mathcal{M}^t}.$$

The left side of the previous inequality is exactly the number of edges of  $M_{t+1}$  not covered by  $\mathcal{M}^t$ , while the right side is  $\frac{t}{2t+1}$  times the number of edges not covered by  $\mathcal{M}^t$ . Denoting by  $a_t$  the fraction of edges of  $E(G)$  covered by  $\mathcal{M}^t$ , we obtain

$$\begin{aligned} m_{t+1} - a_t &\geq a_{t+1} - a_t = (1 - a_t) \cdot \frac{t}{2t+1}, \\ m_{t+1} &\geq (1 - a_t) \cdot \frac{t}{2t+1} + a_t = a_{t+1} \end{aligned}$$

and the assertion follows.  $\square$

By a direct calculation  $a_5 = \frac{215}{231}$  and the following corollary holds:

**Corollary 3.1.** *Let  $G$  be a 3-graph. There exist five perfect matchings of  $G$  which cover at least  $\lceil \frac{215}{231}|E(G)| \rceil$  edges of  $G$ .*

Furthermore, we obtain the following corollary by the proof of Theorem 3.1.

**Corollary 3.2.** *Let  $G$  be a 3-graph. There exist  $t$  perfect matchings of  $G$  with no  $(2t+1)$ -cut in their intersection.*

## 4 A bound for the size of a $t$ -coverable 3-graph

As remarked in [9], it is unknown whether  $m_t = 1$  for any  $t \geq 5$  and the best known result in this direction is the following: for any 3-graph of size  $E$ ,  $m_t = 1$  for  $t$  larger than  $\log_{\frac{3}{2}}(E)$ . In other words the edge-set of each 3-graph of size less than  $(\frac{3}{2})^t$  can be covered by  $t$  perfect matchings.

The following improvement of that bound is an easy consequence of Theorem 3.1.

**Theorem 4.1.** *Let  $G$  be a 3-graph of size  $E$ , such that  $E < \frac{2^t}{\sqrt{t}}$ . Then there exists a covering of  $G$  by  $t$  perfect matchings.*

*Proof.* Let  $G$  be a 3-graph of size  $E$ . It is trivial that if  $Em_t > E - 1$ , that is  $E < \frac{1}{1-m_t}$ , then there exists a covering of  $G$  by  $t$  perfect matchings. One can verify that the inequality  $\frac{2^t}{\sqrt{t}} < \frac{1}{1-a_t}$  holds, then the assertion follows by Theorem 3.1.  $\square$

## 5 Final Remarks

We would like to recall that the particular cases  $m_2 \geq a_2 = \frac{3}{5}$  and  $m_3 \geq a_3 = \frac{27}{35}$  of Theorem 3.1 were already proved in [5]. Furthermore, it can be immediately checked that the Petersen graph realizes the equality  $m_2 = a_2$ . Mainly for this reason, it is conjectured in [9] that also  $m_3$  and  $m_4$  reach their minimum if  $G$  is the Petersen graph.

It is trivial that a possible counterexample to the Berge-Fulkerson conjecture is a snark: a stronger version of the Berge-Fulkerson conjecture could be that  $m_4(G) = 1$  for each snark but the Petersen graph (see [2]). Essentially the Petersen graph should be the unique obstruction to obtaining a cover with at most four perfect matchings. Following this point of view, Bonvicini and the author prove in a forthcoming paper [1] that a large class of 3-graphs (including the Petersen graph) of order  $2n$  can be covered by at most 4 matchings of size  $n - 1$ .

Another possible future study could be to prove an analogous of Theorem 3.1 for  $r$ -graphs, with  $r > 3$ : Seymour (see [10]) proposed a natural generalization of the Berge-Fulkerson conjecture for this class of graphs and it turns out that his conjecture is equivalent to the assertion that each  $r$ -graph can be covered with at most  $2r - 1$  perfect matchings (see [8]). Which portion of the edges of an  $r$ -graph can be covered with  $t$  perfect matchings?

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